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Thickness Stretch Vibrations of Piezoelectric Ceramic Plates for Resonator Applications

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Abstract-**The thickness stretch vibrations of piezoelectric ceramic plates are analyzed by solving the first-order Mindlin plate equations with finite element method in the two-dimensional domain. The precise resonance frequency and distribution of displacements are obtained from the analysis in detail. The results based on the two-dimensional solutions are more important particularly in the evaluation of the energy-trapping feature of ceramic resonators because the accurate mapping of the displacements including the vital thickness stretch mode is of great practical interests. We start from the first-order Mindlin plate theory for PZT type ceramic plates for resonator applications. The cut-off frequency of the thickness stretch vibrations is obtained from the coupled equations. Then these equations are reformatted with the known fundamental resonance frequency and related elastic constants for finite element solutions. With given geometry of a resonator model, the numerical solutions include the resonance frequencies and associated mode shapes are calculated. The thickness stretch vibrations and the associated frequency are of great importance because these results can be directly used for the determination of the frequency of a resonator and the optimal layout of the electrodes for best performance. All these analyses are intended for the direct applications in the design of ceramic resonators. Further considerations of the effects of electrodes, support structures, and other complications can be readily included in current finite element analysis.**

I. INTRODUCTION

It has been a general practice in crystal resonator design process that the vibrations of crystal plate is analyzed with plate theories for high frequency vibrations, particularly with Mindlin plate theory, for many important information like the mechanical vibrations frequency and energy trapping properties. The plate theories specialized for piezoelectric crystal plates have been studied for many years, and they are providing important design tips in the fast shrinking crystal resonators. For instance, the optimal aspect ratios, the ideal electrode configuration, the effect of the support structures, and other related issues can be better addressed from mechanical vibration analysis. In recent applications, the finite element method has been applied with some special treatments to accommodate the fact that there are larger numbers of variables at the extremely high frequency in the thickness-shear (TSh) vibration mode, and the presence of electrodes and electrical variables introduced further challenges in the powerful solution method [1]. The results based on the finite analysis have been finding direct applications in the design process, and further improvements in the research work include three-dimensional solution technique and consideration of electrode and support structures including the packaging materials and manufacturing process [2]. There are many literatures on the plate theories and applications with emphasis on crystal resonators [3].

Piezoelectric ceramic resonators by nature are very close to crystal resonators because the mechanical vibrations are utilized to generate the stable frequency source. The apparent differences are in the frequency range, size of devices, and of course, more importantly, the performance in terms of electrical properties for circuits. However, we shall concentrate on the mechanical vibrations analysis, because it is not well studied in comparison to the crystal devices. Giving the fact that the design of ceramic resonators can also be optimized and benefited from plate vibrations analysis just like crystal resonators, we shall look into the direct applications of piezoelectric plate theories that have been familiar and well utilized in crystal resonator analysis. The essential information from the vibration analysis, like the fundamental frequency, mode shape, coupling, and others will be presented for design applications.

II. VIBRATION MODES AND EQUATIONS

Unlike the crystal resonators, which use the thickness-shear vibration modes, the primary vibration mode in ceramic resonators is the thickness stretch mode. As usual, the thickness stretch mode is coupled with other modes, here are the length and width extensional modes, and the analysis should be done with the system of vibration equations of anisotropic materials, which usually are the three-dimensional equations of elasticity. On the other hand, these equations can be simplified so the strongly coupled equations can be grouped so the analysis will be possible. This effort is well represented by the higher-order plate theory of Mindlin [4], which expands the displacements of a plate in the thickness coordinates and the integration changes the three-dimensional problem to a two-dimensional one. The details of the derivation will not be given here but can be found in Reference [4]. For resonator applications, the thickness-shear mode and other coupled modes have been thoroughly investigated, but these equations, or their variations based on other considerations, can be equally applicable to ceramic resonators. In this study, we apply Mindlin plate theory to the thickness stretch vibrations with coupled extensional modes, which will be shown in next sections.

For a ceramic resonator model as shown in Fig. 1, with $2b$ as the thickness of the plate, for ceramic materials, through the consideration of the PZT-4 materials properties, we have the three vibration equations of thickness stretch and extensional modes [4] as

$$c_{11}u_{1,11}^{(0)}+c_{12}u_{2,21}^{(0)}+c_{13}u_{3,1}^{(1)}+c_{66}\left(u_{1,22}^{(0)}+u_{2,12}^{(0)}\right)+\frac{1}{2b}F_1^{(0)}=\rho\ddot{u}_1^{(0)},$$

$$c_{66}\left(u_{1,21}^{(0)}+u_{2,11}^{(0)}\right)+c_{21}u_{1,12}^{(0)}+c_{22}u_{2,22}^{(0)}+c_{23}u_{3,2}^{(1)}+\frac{1}{2b}F_2^{(0)}=\rho\ddot{u}_2^{(0)},$$

$$c_{55}u_{3,11}^{(1)}+c_{44}u_{3,22}^{(1)}-\frac{3}{b^2}\left(c_{31}u_{1,1}^{(0)}+c_{32}u_{2,2}^{(0)}+c_{33}u_3^{(1)}\right)+\frac{3}{2b^3}F_3^{(1)}=\frac{12}{\pi^2}\rho\ddot{u}_3^{(1)}. \tag{1}$$

where $u_1^{(0)}$, $u_2^{(0)}$, and $u_3^{(1)}$ represent the length extension, width extension, and thickness stretch modes. The elastic constants are denoted as c_{ij} and the density is ρ.

From (1), we can easily find the fundamental thickness stretch vibration frequency is

$$\omega_0=\frac{\pi}{2b}\sqrt{\frac{c_{33}}{\rho}}, \tag{2}$$

and this can be used in the initial design of ceramic resonators for the selection of ceramic plate thickness.

Analytical solutions for (1) with the boundary conditions for resonators are impossible to obtain, and other techniques have to be employed for useful results. One frequently used method is to obtain the approximate solutions based on the straight-crested wave assumptions for plates with a large aspect ratio in one direction. This simple but practical method has been widely used and many useful results have been extracted for design improvements [5]. We can certainly use this method for one-dimensional solutions of (1), but we shall conduct the finite element analysis for two-dimensional solutions in this study with a sophisticated software tool.

III. Finite Element Analysis

The finite element method as a general technique for partial differential equations in engineering applications has been popular and well studied and utilized in many engineering fields including the analysis of crystal resonators [1, 2]. The traditional approach is to start with the differential equations by applying discretizing methods to generate linear equations, thus effectively changing the differential equations to a large linear system. Since there are sophisticated methods for the linear equation manipulations and solutions, the important step in the analysis is usually the specific method for the differential equation discretization. The applications and details of these methods can be found in many finite analysis implementations specific to crystal resonators [1]. However, there is sophisticated software to handle even the basic steps like the decartelization based on the standard procedure. Then, with the resulted linear equations, the solutions, be the eigenvalues for free vibration problems or the electrical parameters for driven problems, can be easily found with readily available mathematical tools. For this part, we found Femlab [6], in conjunction to Matlab, is one of the effective tools based on symbolic computation for this purpose, and we utilize it in this study.

For our free vibration analysis of the ceramic plate, we set the surface traction terms $F_j^{(0)}(j=1,2,3)$ to zero, and the equations will be written in the standard form for the software procedure. The mode shape of the fundamental thickness stretch mode and the vibration frequency in normalized value are computed.

The PZT-4 piezoelectric ceramic, elastic constants used in this study are

$$c=\begin{bmatrix}13.90 & 7.43 & 7.78 & & & \\ & 11.50 & 7.43 & & & \\ & & 13.90 & & & \\ & & & 2.56 & & \\ & & & & 2.56 & \\ & & & & & 2.56\end{bmatrix}\times 10^{10}\ \mathrm{N/m^2}, \tag{3}$$

with the density of 7500 $\mathrm{kg/m^3}$. The aspect ratios of length to thickness and width to thickness are 30 and 20, respectively.

We plot the mode shape for the fundamental thickness stretch vibrations and the extensional modes at the resonance frequency in Fig. 2.

Although the finite element analysis can be made much simpler with the Femlab software, the analysis process and results are somewhat different from the available traditional applications. As a result, the utilization of the analysis will have to be modified to reflect the changes. For instance, the frequency spectra, or the frequency changes versus aspect ratios, can no longer be easily obtained due to the limitations of the software. Consequently, the analytical results should be visualized directly to see the changes or improvements in a more efficient manner. On the other hand, the changes on the design now can be made much easily. It is apparent that the trade off between the current software capability and traditional way of analysis should be considered in the applications for the maximum benefits.

IV. Conclusions

The convenient software based on symbolic approach has been used in the finite element analysis of ceramic resonators and their design process. The simplified finite element analysis enables engineers to take the advantage of vibration analysis in resonator design without investing extensive efforts. The predicted vibration mode shapes and frequencies are essential in the optimal selection of plate aspect ratios, electrode configurations, supporting structures, and other parameters. The precise analysis with finite element tools will be important in the further optimization of the ceramic resonators in production and design.

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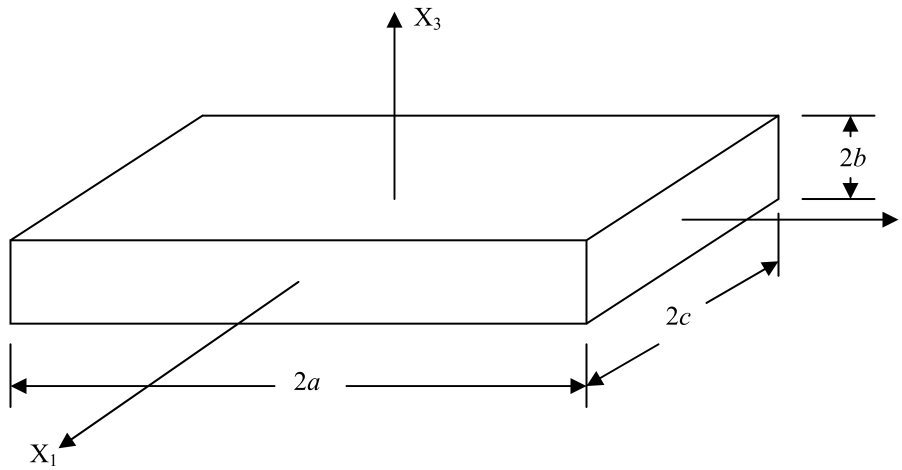

Fig. 1 Physical model of a ceramic plate

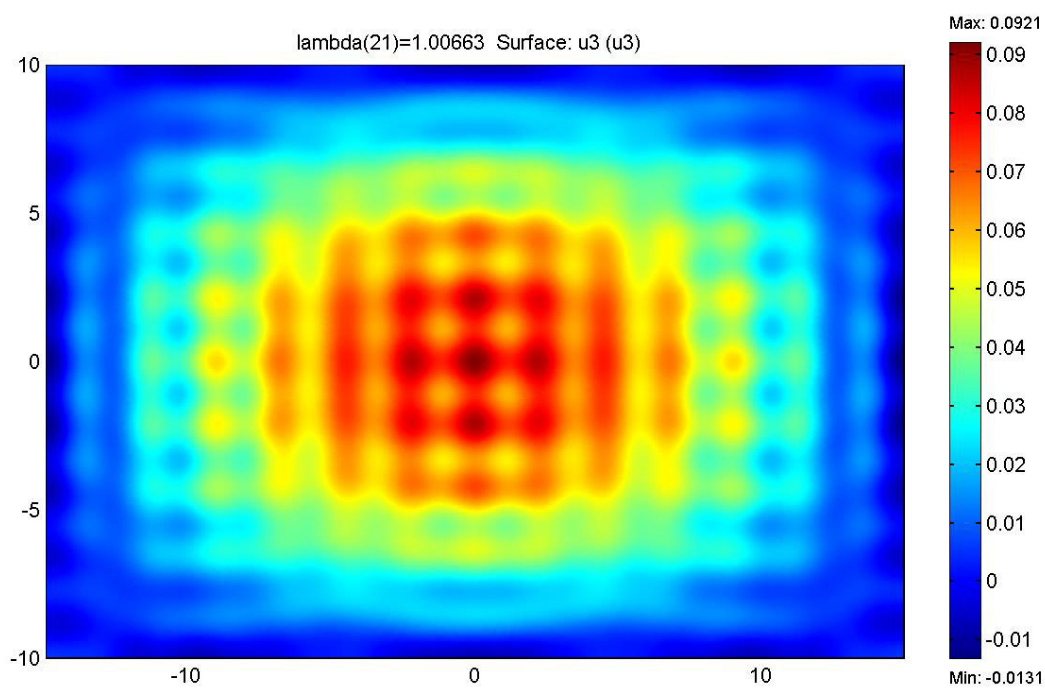

Fig. 2 Thickness stretch vibration of piezoelectric ceramic plate at normalized resonance frequency 1.00663